Dedicated to Vladimir Mikhaylovich Tikhomirov

ON ONE ESTIMATE, CONNECTED WITH THE STABILIZATION OF NORMAL PARABOLIC EQUATION BY STARTING CONTROL

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Abstract. After brief revision of the facts, concerning semilinear parabolic equations of normal type and their nonlocal stabilization by starting control we provide a simplification of the proof of the lower bound for one functional of the solution to heat equation with initial condition of a special type. This bound is essential to prove the nonlocal stabilization of equations of normal type. The simplification presented is required for further development of nonlocal stabilization theory.

Introduction

This work is devoted to the development of the nonlocal stabilization theory for the equations of normal type by starting control. Semilinear normal parabolic equations have been derived in [4]-[8] to understand the structure of the solutions to the thee-dimensional Helmholtz system, describing the curl ω of the velocity vector field v for the viscous fluid flow, and to other similar equations better.

Let us explain the importance of such a study. Velocity v along with pressure p is described by three-dimensional Navier-Stockes system. Since v satisfies the energy estimate, it is possible to prove that there exists a weak solution of the Navier-Stockes system, but existence proof of a strong solution (proven to be unique) is impeded because the curl $\operatorname{rot} v = \omega$ does not satisfy the energy estimate. [1] The latter is due to the fact that the image $B(\omega)$ of the nonlinear operator B, generated by non-linear members of Helmholtz equation, is not orthogonal to vector ω, i.e. it contains component $\Phi\omega$, collinear to ω.

If the non-linear members of Helmholtz system that define the $B(\omega)$ operator, are substituted by members that define the component $\Phi\omega$ collinear to vector ω, then the resulting system of equations is called (by definition) a normal parabolic equation, corresponding to Helmholtz system. The general plan of studying normal parabolic equation (NPE) consists in describing the structure of dynamics, generated by NPE, and in proving the possibility of nonlocal stabilization of NPE by a suitable control. Further we intend to use the results obtained for NPE to study the initial Helmholtz system. The first step is to study the NPE corresponding

Date: 29th of May, 2014.

1991 *Mathematics Subject Classification.* Primary 37D10, 40H05; Secondary 35K57, 35B38.

Key words and phrases. equations of normal type, stabilization by starting control.

The work has been fulfilled by RAS program "Theoretical problems of modern mathematics"', project "'Optimization of numerical algorithms of Mathematical Physics problems"'. Author was supported in part by RFBI Grant #12-01-00262a.

[1] In two-dimensional case ω satisfies the energy estimate, which allowed V.I. Yudovitch [16] to prove the existence of a strong solution even for Euler equations of ideal incompressible liquid.

to the Burgers equation. So far we only consider the case of NPE with periodic boundary condition

At the moment the structure of the dynamical flow corresponding to NPE has been studied for NPE, generated not only by Burgers equation ([4],[5]), but also by Helmholtz system ([6],[7]). Nevertheless the problem of nonlocal stabilization by starting control supported on a given subdomain has been studied only for NPE connected with Burgers equation ([8]). Namely, it was proved that the NPE with arbitrary initial condition y_0 can be stabilized by starting control in the form $u(x) = \lambda u_0(x)$, where λ is some constant, depending on y_0, and u_0 is a universal function, depending only on a given arbitrary subinterval $(a, b) \subset [0, 2\pi)$, which contains the support of control u_0. The following estimate is essential for the proof of the stabilization result:

$$\int_0^{2\pi} S^3(t, x; u_0)dx \geq \beta e^{-6t} \qquad \forall t > 0 \tag{0.1}$$

where $S(t, x; u_0)$ is the solution of the heat equation with initial condition u_0 and $\beta > 0$ is some constant.

We should note that the proof of the estimate (0.1), derived in [8], was so complicated, that its generalization of a NPE connected with three-dimensional Helmholz system is highly problematic. [2] The necessary simplifications of the estimate (0.1) proof constitute the main result of this article.

In the first two sections we remind the definitions and some facts concerning NPE connected with Burgers equation, particularly those concerning nonlocal stabilization. The rest of the sections are devoted to the simplifications of the estimate (0.1) proof.

In conclusion let us note that there exists extensive literature on the local stabilization of hydrodynamic equations in the neighborhood of a stationary point (see for example, [9], [1], [14], [3], [15], [10] , as well as literature, listed in the review [12]). The number of works devoted to nonlocal stabilization as well as to nonlocal exact controllability is much smaller (see for example, [2], [11], [13]).

1. Semilinear parabolic equation of normal type

Lower we remind some basic facts concerning NPE: its derivation, explicit formula for its solution, structure of dynamical flow corresponding to NPE.

1.1. **Derivation of the normal parabolic equation (NPE).** In this paper we only consider the NPE corresponding to Burgers equation. Let us consider the Burgers equation:

$$\partial_t v(t, x) - \partial_{xx} v(t, x) - \partial_x v^2(t, x) = 0 \tag{1.1}$$

with periodic boundary condition and initial data

$$v(t, x + 2\pi) = v(t, x), \qquad v(t, x)|_{t=0} = v_0(x) \tag{1.2}$$

As is well known, the orthogonality in $L_2(0, 2\pi)$ of the quadratic member $\partial_x v^2(t, x)$ from equation (3.1) to $v(t, x)$ leads to energy estimate for the solutions of Burgers equation:

$$\int_0^{2\pi} v^2(t, x)dx + 2\int_0^t \int_0^{2\pi} (\partial_x v(t, x))^2 dxdt \leq \int_0^{2\pi} v_0^2(x)dx$$

[2]The generalization of the result about nonlocal start control stabilization on the case of a NPE connected with three-dimensional Helmholz system is currently our major goal.

But an analogous relation does not hold for function $\partial_x v(t,x)$. Indeed, after differentiating (3.1) by x, we get

$$\partial_t v_x(t,x) - \partial_{xx} v_x(t,x) - B(v, v_x) = 0, \tag{1.3}$$

where $v_x = \partial v/\partial x$, and

$$B(v, v_x) = 2v_x^2 + 2v\partial_x v_x \tag{1.4}$$

Multiplying (1.4) scalarly in $L_2(T_1)$ by v_x, where $T_1 = \mathbb{R}/2\pi\mathbb{Z}$ is a circle, and integrating by parts, we get that

$$\int_0^{2\pi} B(v, v_x) v_x dx = \int_0^{2\pi} (2v_x^3 + 2vv_x\partial_x v_x) dx = \int_0^{2\pi} v_x^3 dx \neq 0 \tag{1.5}$$

Let us introduce an important space

$$L_2^0(T_1) = \{v(x) \in L_2(T_1) : \int_0^{2\pi} v(x)dx = 0\}, \tag{1.6}$$

and decompose operator $B(v, v_x)$ in this space into a normal and tangent components:

$$B(v, v_x) = B_n(v, v_x) + B_\tau(v, v_x) \tag{1.7}$$

where $B_n(v, v_x) = \Phi(v, v_x)v_x$, and $\Phi(v, v_x)$ is a functional, while vector $B_\tau(v, v_x)$ is orthogonal to vector v_x in $L_2^0(T_1)$:

$$\int_0^{2\pi} B_\tau(v, v_x) v_x dx = 0 \tag{1.8}$$

To define $\Phi(v, v_x)$, let us substitute (1.7) into (1.5) and use (1.8). As the result, we get:

$$\int_0^{2\pi} v_x^3 dx = \int_0^{2\pi} \Phi(v, v_x) v_x^2 dx = \Phi(v, v_x) \int_0^{2\pi} v_x^2 dx$$

Therefore, functional Φ does not depend on v, but only depends on v_x, and for $v_x \neq 0$

$$\Phi(v_x) = \int_0^{2\pi} v_x^3 dx \Big/ \int_0^{2\pi} v_x^2 dx$$

Obviously, Φ can be by continuity defined as zero for $v_x \equiv 0$, i.e. $\Phi(0) = 0$.

Substituting term $B(v, v_x)$ in equation (1.3) with $\Phi(v_x)v_x$, and using notation $v_x = y$, we get:

$$\partial_t y(t,x) - \partial_{xx} y(t,x) - \Phi(y)y = 0, \tag{1.9}$$

where

$$\Phi(y) = \begin{cases} \int_0^{2\pi} y(x)^3 dx \Big/ \int_0^{2\pi} y(x)^2 dx, & y \neq 0 \\ 0, & y \equiv 0 \end{cases} \tag{1.10}$$

Equation (1.9) is called semilinear parabolic equation of the normal type or normal parabolic equation (NPE). We shall consider this equation with the periodic boundary condition

$$y(t, x + 2\pi) = y(t, x) \tag{1.11}$$

and initial condition

$$y(t,x)|_{t=0} = y_0(x) \tag{1.12}$$

Existence and uniqueness theorems for the problems (1.9)-(1.12) in corresponding function spaces have been proved similarly to the case of the NPE corresponding to three-dimensional Helmholtz equation ([7]). For briefness we will not formulate these theorems but consider the property of NPE which is essential for studying equations of this type.

1.2. **Explicit formula for NPE solution.** The following lemma is true:

Lemma 1.1. *Let $S(t, x; y_0)$ be the solution of the heat equation*

$$\partial_t S - \partial_{xx} S = 0, \qquad S|_{t=0} = y_0(x) \tag{1.13}$$

with periodic boundary condition. Then the solution of the problem (1.9)-(1.12) can be written as

$$y(t, x; y_0) = \frac{S(t, x; y_0)}{1 - \int_0^t \Phi(S(\tau, x; y_0)) d\tau} \tag{1.14}$$

This lemma has been proved in [5]. Its proof consists in substitution the given formula into (1.9) and direct checking.

1.3. **Structure of dynamic flow of NPE.** Let us recall the main feature of the dynamical flow corresponding to problem (1.9)- (1.12). Namely, we shall decompose the phase space $L_2^0(T_1)$ of the dynamical system into three sets, where behavior of dynamical flow differs significantly.

Definition 1.1. *The set $M_- \subset L_2^0(T_1)$ of initial conditions y_0 such that the solution $y(t, x; y_0) \in H^{1,2(-1)}(Q)$ of problem (1.9), (1.12) exists and satisfies inequality*

$$\|y(t, \cdot; y_0)\|_0 \leq \alpha \|y_0\|_0 e^{-t} \qquad \forall t > 0 \tag{1.15}$$

is called the set of stability. Here $\|\cdot\|$ is the norm of the phase space $L_2^0(T_1)$, $\alpha = \alpha(y_0) > 1$ is a certain fixed number dependent on y_0.

Definition 1.2. *The set $M_+ \subset L_2^0(T_1)$ of initial conditions y_0 from (1.9), (1.12) such that corresponding solution $y(t, x; y_0)$ exists only on a finite interval $t \in (0, t_0)$ with $t_0 > 0$ depending on y_0, and blows up at $t = t_0$* [3] *is called the set of explosions.*

By virtue of formula (1.14) for solution $y(t, x; y_0)$

$$M_+ = \{y_0 \in L_2^0(T_1) : \ \exists t_0 > 0 \int_0^{t_0} \Phi(S(\tau, \cdot; y_0)) d\tau = 1\} \tag{1.16}$$

The minimal magnitude from the set of numbers $\{t_0\}$ satisfying (1.16) is called the time of explosion.

Definition 1.3. *The collection $M_g \subset H^0(T_1)$ of initial conditions y_0 from (1.9), (1.12) such that corresponding solution $y(t, x; y_0)$ exists on infinite time interval $t \in \mathbb{R}_+$ and $\|y(t, \cdot; y_0)\|_0 \to \infty$ as $t \to \infty$ is called the set of growth.*

The following statement is true:

Theorem 1.1. *The sets of stability, explosions and growth are not empty:*

$$M_- \neq \emptyset, \quad M_+ \neq \emptyset, \quad M_g \neq \emptyset$$

Moreover,

$$M_- \cup M_+ \cup M_g = L_2^0(T_1)$$

[3] i.e. $y(t, x; y_0) \in H^{1,2(-1)}(Q_{t_0-\varepsilon})$ for arbitrary small enough ε, and $\|y(t, \cdot; y_0)\|_0 \to \infty$ as $t \to t_0$.

The theorem has been proved in [5], [7]. Let us note, that in [5], [7] not only algebraic, but also geometric properties of sets M_-, M_+, M_g were studied. We will not state many of these results here, for they will not be required below. [4] We need the theorem 1.1 to show the intensionality of the theorem on NPE stabilization by starting control.

2. Stabilization of the solution of NPE via start control

In this paragraph we formulate the basic theorem of stabilization of NPE, the lower bound theorem necessary for its proof, as well as the scheme of the first step of the proof of this lower bound theorem.

2.1. **Formulation of the main stabilization result.** Let us consider the boundary problem for NPE (1.9)-(1.12), where the initial condition (1.12) is changed for

$$y(t,x)|_{t=0} = y_0(x) + v(x), \quad x \in T_1 = \mathbb{R}/2\pi\mathbb{Z}, \tag{2.1}$$

where $y_0(x)$ is the given initial condition, and $v(x)$ is the starting control. It is assumed that control $v(x)$ is supported on a subinterval of a given segment $[a, b] \subset T_1$:

$$\operatorname{supp} v \subset [a,b] \tag{2.2}$$

The stabilization problem is formulated as follows:

For a given $y_0(x) \in L_2^0(T_1)$ find such a control $v \in L_2^0(T_1)$, satisfying (2.2), that the solution $y(t,x;y_0+v)$ of problem (1.9),(1.11),(2.1) is defined for all $t > 0$ and satisfies the estimate

$$\|y(t,\cdot;y_0+v)\| \le \alpha \|y_0+v\| e^{-t} \quad \forall t > 0 \tag{2.3}$$

for some constant $\alpha > 1$.

Let us note that the stated problem is rich in content only for $y_0 \in M_+$ or $y_0 \in M_g$, as for $y_0 \in M_-$ by virtue of (1.15) it is enough to set $v \equiv 0$ for stabilization.

The following theorem is true:

Theorem 2.1. *For any given $y_0(x) \in M_+ \cup M_g$ there exists a control $v \in L_2^0(T_1)$, satisfying (2.2), that provides the solution of the stabilization problem.*

2.2. **Formulation of the key result on the estimate.** By substitution $\tilde{x} = x - \frac{a+b}{2}$ in (1.9),(1.11),(2.1) the problem of stabilization can be reduced to the case when condition (2.2) on the support of control v has the form

$$\operatorname{supp} v(x) \subset [-\rho,\rho], \qquad 0 < \rho < \pi \tag{2.4}$$

with $\rho = (b-a)/2$. Below we shall write the circle T_1 as a segment $[-\pi,\pi]$ with its ends identified.

In fact we suggest a universal starting control, that can be defined up to a constant factor as follows. For a given $\rho \in (0,\pi)$ we choose such a $p \in \mathbb{N}$ that

$$\frac{\pi}{2p} < \rho \tag{2.5}$$

[4] The most complete results on geometrical structure of sets M_-, M_+, M_g are obtained in [7]in the case of NPE corresponding 3D Helmholtz equations. The same results in the case of NPE corresponding to the Burgers equations can be obtained absolutely similarly.

and define the characteristic function of the interval $(-\frac{\pi}{2p}, \frac{\pi}{2p})$ as

$$\chi(x) = \begin{cases} 1, & |x| \leq \frac{\pi}{2p} \\ 0, & \frac{\pi}{2p} < |x| \leq \pi \end{cases} \tag{2.6}$$

We search for the control in the form $v(x) = \lambda u(x)$ where λ is some constant, and function $u(x)$ has the following form:

$$u(x) = \chi(x)(\cos 2px + \cos 4px) \tag{2.7}$$

Obviously,

$$u(x) \in L_2^0(T_1), \qquad \operatorname{supp} u \subset [-\rho, \rho] \tag{2.8}$$

Let us consider the boundary problem for the heat equation

$$\partial_t S(t,x) - \partial_{xx} S(t,x) = 0, \qquad S(t,x)|_{t=0} = u(x) \tag{2.9}$$

with periodic boundary condition. The following estimate is true for the solution of this problem:

Theorem 2.2. *Let $S(t,x,u)$ be the solution of the boundary problem (2.9) with initial condtion $u(x)$, defined in (2.7). Then*

$$\int_{-\pi}^{\pi} S^3(t,x;u)dx \geq \beta e^{-6t} \qquad \forall\, t \geq 0, \tag{2.10}$$

where β is some positive constant.

Let us note, that embedding $u \in L_2^0(T_1)$ implies the equality $\int_{-\pi}^{\pi} u(x)dx = 0$ (see (1.6)), which considerably complicates the proof of the theorem. Theorem 2.2 was proved in [8]. But some parts of that proof are too complicated to be generalized on the case of NPE corresponding to the three-dimensional Helmholtz systems. Note that the stabilization of the NPE corresponding to the three-dimensional Helmholtz systems, is currently our main goal.

The major purpose of this paper is the simplification of the proof of the theorem 2.2 to a satisfactory level. In the remaining part of this section we recall the draft for the first step of the theorem 2.2 proof. Further sections are devoted to the simplification of the main part of this theorem proof.

2.3. **Draft for the first step of the theorem 2.2 proof.** Periodic function (2.7) can be decomposed into Fourier series

$$u(x) = \sum_{r \in \mathbb{Z}} \widehat{u}_k e^{ikx},$$

and first of all we have to calculate the Fourier coefficients of this function. As it was shown in [8]

$$\widehat{u}_{2p} = \widehat{u}_{-2p} = \widehat{u}_{4p} = \widehat{u}_{-4p} = 1/(4p) \tag{2.11}$$

and for $k \neq \pm 2p \pm 4p$

$$\widehat{u}_k = \frac{k}{\pi} \sin \frac{k\pi}{2p} \sum_{m=1}^{2} \frac{(-1)^m}{k^2 - (2pm)^2} = \frac{12p^2 k \sin \frac{k\pi}{2p}}{\pi(k^2 - 16p^2)(k^2 - 4p^2)} \tag{2.12}$$

Fourier decomposition of the solution $S(t,x;u)$ to the boundary value problem (2.9) has the following form:

$$S(t,x,u) = \sum_{k \in \mathbb{Z}} e^{-k^2 t} \widehat{u}_k e^{ikx}$$

where $\widehat{u}_k$ are the coefficients defined in (2.11),(2.12).

By virtue of (2.11),(2.12) the solution $S(t,x,u)$ to problem (2.9) can be rewritten as

$$S(t,x,u) = S_1(t,x,u) + S_2(t,x,u), \tag{2.13}$$

where

$$S_1(t,x,u) = \frac{1}{4p}\sum_{m=1}^{2}(e^{2pmix} + e^{-2pmix})e^{-(2pm)^2 t} \tag{2.14}$$

$$S_2(t,x,u) = \frac{12p^2}{\pi}\sum_{\substack{k\in\mathbb{Z}\\ k\notin\{\pm 2p,\pm 4p\}}} \frac{k\sin\frac{k\pi}{2p}}{(k^2-16p^2)(k^2-4p^2)}e^{ikx-k^2 t} \tag{2.15}$$

Decomposition (2.13) implies that

$$\begin{aligned}&\int_{-\pi}^{\pi} S^3(t,x,u)dx\\ &= \int_{-\pi}^{\pi}(S_1^3(t,x,u) + 3S_1^2(t,x,u)S_2(t,x,u) + 3S_1(t,x,u)S_2^2(t,x,u) + S_2^3(t,x,u))dx\end{aligned} \tag{2.16}$$

and establishing of (2.10) is reduced to the proof of non-negativeness of the terms in (2.16). The non-negativeness of the integrals connected with the first three terms in the right-hand side of (2.16) has been proved in [8], and the proof for the first two components is very simple.

The integral, connected with the fourth term, can be rewritten as

$$\int_{-\pi}^{\pi} S_2^3(t,x,u)dx = \frac{2^7 3^3 p^6}{\pi^2}\sum_{\substack{k,m\in\mathbb{Z}\setminus\{\pm 2p,\pm 4p\}\\ k+m\notin\{\pm 2p,\pm 4p\}}} B(k)B(m)B(k+m)e^{-(k^2+m^2+(k+m)^2)t}, \tag{2.17}$$

where

$$B(k) = \frac{k\sin\frac{\pi k}{2p}}{(k^2-16p^2)(k^2-4p^2)} \tag{2.18}$$

Since $B(k)$ is an even function of k, one can show (see [8]), that equality (2.17) can be rewritten as

$$\int_{-\pi}^{\pi} S_2^3(t,x,u)dx = \frac{2^8 3^4 p^6}{\pi^2}J_1, \tag{2.19}$$

where

$$J_1 = \sum_{\substack{k,m\in\mathbb{N}\setminus\{2p,4p\}\\ k+m\notin\{2p,4p\}}} B(k)B(m)B(k+m)e^{-(k^2+m^2+(k+m)^2)t} \tag{2.20}$$

Therefore, the proof of the main theorem 2.2 is reduced to the proof of the following statement:

Theorem 2.3. *The following inequality is true:*

$$J_1 \geq \beta e^{-6t} \qquad \forall t > 0, \tag{2.21}$$

where J_1 is the functional (2.20) and β is some positive constant.

The rest of the article is devoted to the proof of this theorem.

Let us first remind one result from [8].

3. Distribution of signs for summands from the sum J_1

Let us now determine how the signs of summands in J_1 are distributed. We will need the following lemma, directly following from definition (2.18) of function $B(k)$:

Lemma 3.1. *Let $B(k), k \in \mathbb{N} \setminus \{2p, 4p\}$ be the function, defined in (2.18). Then*

$$\begin{aligned} &i)\ B(k) = 0 \quad \text{for} \quad k = 2pl \quad \text{where} \quad l \in \mathbb{N}, l \geq 3 \\ &ii)\ B(k) > 0 \quad \text{for} \quad k \in (0, 2p) \cup (2p, 4p) \cup \{\cup_{l \in \mathbb{N}} (4lp, (4l+2)p) \\ &iii)\ B(k) < 0 \quad \text{for} \quad k \in \cup_{l \in \mathbb{N}} ((4l+2)p, 4(l+1)p) \end{aligned}$$

The next statement follows directly from Lemma 3.1:

Lemma 3.2. *The signs of function $B(k)B(m)B(k+m)$ from (2.20) are distributed as follows:*
i)

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \begin{cases} +, & \text{if } k+m < 2p(a+b+1) \\ -, & \text{if } k+m > 2p(a+b+1) \end{cases} \tag{3.1}$$

in each square $\{(k,m) \in (2pa, 2p(a+1)) \times (2pb, 2p(b+1))\}$, where $a, b \in \mathbb{N} \setminus \{1\}$
ii)

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \begin{cases} +, & \text{if } k+m < 2p(a+1) \\ -, & \text{if } k+m > 2p(a+1) \end{cases} \tag{3.2}$$

in every set $\{(k,m) \in (2pa, 2p(a+1)) \times (0, 2p) \cup (0, 2p) \times (2pa, 2p(a+1))\}$, $a \in \mathbb{N} \setminus \{1\}$
iii)

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \begin{cases} -, & \text{if } k+m < 2p(a+2) \\ +, & \text{if } k+m > 2p(a+2) \end{cases} \tag{3.3}$$

in every square $\{(k,m) \in (2pa, 2p(a+1)) \times (2p, 4p) \cup (2p, 4p) \times (2pa, 2p(a+1))\}$ where $a \in \mathbb{N} \setminus \{1\}$
iv)

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \begin{cases} +, & \text{if } k+m < 6 \\ -, & \text{if } k+m > 6p \end{cases} \tag{3.4}$$

for $(k,m) \in \{(0, 4p) \times (0, 4p)\}$

Proof. Let us check statement i). Since $k \in (2pa, 2p(a+1))$, $m \in (2pb, 2p(b+1))$, and $a \geq 2, b \geq 2$, relation (2.18) implies that

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \operatorname{sign}\left(\sin\frac{\pi k}{2p}\sin\frac{\pi m}{2p}\sin\frac{\pi(k+m)}{2p}\right) \tag{3.5}$$

Obviously, $\operatorname{sign}(\sin\frac{\pi k}{2p}) = \operatorname{sign}(-1)^a$ for $k \in (2pa, 2p(a+1))$. In the case in question either $2p(a+b) < k+m < 2p(a+b+1)$, or $2p(a+b+1) < k+m < 2p(a+b+2)$. In the first case

$$\operatorname{sign}\left(\sin\frac{\pi k}{2p}\sin\frac{\pi m}{2p}\sin\frac{\pi(k+m)}{2p}\right) = \operatorname{sign}\left((-1)^{a+b}(-1)^{a+b}\right)$$

In the second case

$$\operatorname{sign}\left(\sin\frac{\pi k}{2p}\sin\frac{\pi m}{2p}\sin\frac{\pi(k+m)}{2p}\right) = \operatorname{sign}\left((-1)^{2(a+b)+1}\right).$$

This, in virtue of (3.5), implies (3.1).

ii) If $(k,m) \in (2pa, 2p(a+1)) \times (0,2p)$, then (2.18) and point ii) of Lemma 3.1 imply relations

$$\operatorname{sign}(B(k)B(m)B(k+m)) = \operatorname{sign}(B(k)B(k+m)) = \operatorname{sign}\left(\sin\frac{\pi k}{2p}\sin\frac{\pi(k+m)}{2p}\right)$$

and we get for $2pa < k+m < 2p(a+1)$ that $\operatorname{sign}\left(\sin\frac{\pi k}{2p}\sin\frac{\pi(k+m)}{2p}\right) = \operatorname{sign}\left((-1)^{2a}\right)$. If $2p(a+1) < k+m < 2p(a+2)$, then $\operatorname{sign}\left(\sin\frac{\pi k}{2p}\sin\frac{\pi(k+m)}{2p}\right) = \operatorname{sign}\left((-1)^{2a+1}\right)$. This proves (3.2). The other case of ii), as well as points iii) and iv) of Lemma 3.2 can be checked similarly. □

Let us denote by $J_1(i), J_1(ii), J_1(iii), J_1(iv)$ parts of sum J_1, whose summands are described in points i),ii),iii),iv) of Lemma 3.2 correspondingly and demonstrate the positiveness of these sums.

4. Positiveness of sums $J_1(i)$ and $J_1(iv)$

4.1. **Positiveness of $J_1(i)$.** First let us prove the following auxiliary statement:

Lemma 4.1. *Let $p \in \mathbb{N}$. Then the function*

$$D(x) = \frac{x}{(x^2-16p^2)(x^2-4p^2)} \tag{4.1}$$

decreases monotonically for $x > 4p$.

Proof. It is easy to see that for $x > 4p$

$$D(x) = \frac{1}{24p^2}\left(\frac{1}{x-4p} + \frac{1}{x+4p} - \frac{1}{x-2p} - \frac{1}{x+2p}\right) \tag{4.2}$$

Therefore,

$$D'(x) = -\frac{1}{24p^2}\left[\left(\frac{1}{(x-4p)^2} - \frac{1}{(x-2p)^2}\right) + \left(\frac{1}{(x+4p)^2} - \frac{1}{(x+2p)^2}\right)\right] < 0 \tag{4.3}$$

□

Let (k,m) be the point in $\mathbb{N}\times\mathbb{N}$. Let us denote the set of all points from $\mathbb{N}\times\mathbb{N}$ belonging to the triangle with vertices at points $(k_1,m_1),(k_2,m_2),(k_3,m_3)$ by $\{(k_1, m_1),(k_2,m_2),(k_3,m_3)\}$, and the part of sum J_1, defined in (2.20), consisting of the summands which belong to triangle $\{(k_1,m_1),(k_2,m_2),(k_3,m_3)\}$ by $J_1(\{(k_1,m_1), (k_2,m_2),(k_3,m_3)\})$.

We prove now the main statement of this section.

Lemma 4.2. *Let $a,b,p \in \mathbb{N},\ a \geq 2, b \geq 2$. Then*

$$\begin{aligned} &J_1(\{(2pa,2pb),(2p(a+1),2pb),(2pa,2p(b+1))\}) \\ &+ J_1(\{(2pa,2p(b+1)),(2p(a+1),2pb),(2p(a+1),2p(b+1))\}) > 0 \end{aligned} \tag{4.4}$$

for every $t \geq 0$.

Proof. Let us substitute variables k,m in (4.4) with variables k_1,m_1 by formulas $k = 2pa+k_1,\ m = 2pb+m_1$, and introduce the following notations:

$$A(k) := B(k)e^{-k^2t} = D(k)e^{-k^2t}\sin\frac{\pi k}{2p} \tag{4.5}$$

where the last equality follows from (2.18),(4.1). Then

$$
\begin{aligned}
&J_1(\{(2pa, 2pb), (2p(a+1), 2pb), (2pa, 2p(b+1))\}) \\
&= \sum_{\substack{k_1, m_1 = 1 \\ k_1 + m_1 < 2p}}^{2p-1} A(2pa + k_1)A(2pb + m_1)A(2p(a+b) + k_1 + m_1)
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
&J_1(\{(2pa, 2p(b+1)), (2p(a+1), 2pb), (2p(a+1), 2p(b+1))\}) \\
&= \sum_{\substack{k_1, m_1 = 1 \\ k_1 + m_1 > 2p}}^{2p-1} A(2pa + k_1)A(2pb + m_1)A(2p(a+b) + k_1 + m_1)
\end{aligned} \tag{4.7}
$$

According to Lemma 3.2, the summands in $J_1(\{(2pa, 2pb), (2p(a+1), 2pb), (2pa, 2p(b+1))\})$ are positive, and the summands in $J_1(\{(2pa, 2p(b+1)), (2p(a+1), 2pb), (2p(a+1), 2p(b+1))\})$ are negative. Let us show, that the absolute values of negative summands are not greater than the absolute values if positive summands. This will prove (4.4)

Let us perform the change of variables $s = 2p - k_1 - m_1$, $k = k_1$ in (4.6), and the change of variables $s = k_1 + m_1 - 2p$, $k = k_1 - s$ in (4.7) and consider the following ratio:

$$
\begin{aligned}
&\frac{|A(2pa + k + s)A(2p(b+1) - k)A(2p(a+b+1) + s)|}{A(2pa + k)A(2p(b+1) - k - s)A(2p(a+b+1) - s)} = \\
&\frac{D(2pa + k + s)D(2p(b+1) - k)D(2p(a+b+1) + s)}{D(2pa + k)D(2p(b+1) - k - s)D(2p(a+b+1) - s)} \cdot e^{-12(a+b+1)ps} < \\
&\frac{D(2pa + k + s)}{D(2pa + k)} \frac{D(2p(b+1) - k)}{D(2p(b+1) - k - s)} \frac{D(2p(a+b+1) + s)}{D(2p(a+b+1) - s)}.
\end{aligned} \tag{4.8}
$$

According to Lemma4.1 all three fractions in the last line of (4.8) are less than one. Therefore, sum (4.4) is positive. □

4.2. **Positiveness of $J_1(iv)$.** Below we prove the statement that implies the positiveness of $J_1(iv)$.

Lemma 4.3. *The following relation holds:*

$$
J_1(\{(2p, 2p), (2p, 4p), (4p, 2p)\}) + J_1(\{(2p, 4p), (4p, 4p), (4p, 2p)\}) \geq 0 \tag{4.9}
$$

Proof. Let us perform the change of variables $(k, m) \to (x, y)$ by formulas $k = 2p + 2xp, m = 2p + 2yp$ in triangle $\{(2p, 2p), (2p, 4p), (4p, 2p)\}$, and by formulas $k = 4p - 2xp, m = 4p - 2yp$ in triangle $\{(2p, 4p), (4p, 4p), (4p, 2p)\}$. We consider the following relation of the summands from the first and second triangle with the same coordinates (x, y)

$$
\frac{|B(2p(2-x))B(2p(2-y))B(2p(4-x-y))|e^{-(2p)^2((2-x)^2+(2-y)^2+(4-x-y)^2)t}}{B(2p(1+x))B(2p(1+y))B(2p(2+x+y))e^{-(2p)^2((1+x)^2+(1+y)^2+(2+x+y)^2)t}}
$$

where $x \in (0, 1), y \in (0, 1), x + y < 1$ and find the lower bound of this expression to prove (4.9). Since the exponent in the numerator is less than the exponent in the denominator for all $(x, y) \in (0, 1) \times (0, 1), x + y < 1$ it is enough to estimate this fraction for $t = 0$. When $t = 0$, this fraction is equal to

$$I(x,y) := \frac{|D(2p(2-x))\sin\frac{\pi}{2p}[2p(2-x)]D(2p(2-y))\sin\frac{\pi}{2p}[2p(2-y)]|}{|D(2p(1+x))\sin\frac{\pi}{2p}[2p(1+x)]D(2p(1+y))\sin\frac{\pi}{2p}[2p(1+y)]|}$$
$$\cdot\frac{|D(2p(4-x-y))\sin\frac{\pi}{2p}[2p(4-x-y)]|}{|D(2p(2+x+y))\sin\frac{\pi}{2p}[2p(2+x+y)]|} = \frac{|\tilde{D}(2-x)\tilde{D}(2-y)\tilde{D}(4-x-y)|}{|\tilde{D}(1+x)\tilde{D}(1+y)\tilde{D}(2+x+y)|} \tag{4.10}$$

where the first equality is the definition of $I(x,y)$ and

$$\tilde{D}(x) = \frac{x}{(x^2-4)(x^2-1)} \tag{4.11}$$

Note that by virtue of (4.11)

$$\begin{aligned} f_1(x) :&= \frac{|\tilde{D}(2-x)|}{|\tilde{D}(1+x)|} = \frac{(2-x)(4-(1+x)^2)((1+x)^2-1)}{(4-(2-x)^2)((2-x)^2-1)(1+x)} \\ &= \frac{(2-x)(1-x)x(3+x)(2+x)}{(4-x)(3-x)(1-x)x(1+x)} = \frac{2-x}{1+x}\cdot\frac{3+x}{4-x}\cdot\frac{2+x}{3-x} \end{aligned} \tag{4.12}$$

$$\begin{aligned} f_2(z) :&= \frac{|\tilde{D}(4-z)|}{|\tilde{D}(2+z)|} = \frac{(4-z)(4-(2+z)^2)((2+z)^2-1)}{(4-(4-z)^2)((4-z)^2-1)(2+z)} \\ &= \frac{(4-z)z(1+z)(3+z)(4+z)}{(6-z)(5-z)(3-z)(2-z)(2+z)} \end{aligned} \tag{4.13}$$

Since

$$f_2(x) = \tilde{f}_2(x)x, \text{ where } \tilde{f}_2(x) = \frac{4-x}{2-x}\cdot\frac{1+x}{2+x}\cdot\frac{1}{6-x}\cdot\frac{3+x}{5-x}\cdot\frac{4+x}{3-x}$$

and every multiplier in the definition of $\tilde{f}_2(x)$ increases,

$$f_2(x) \le \tilde{f}_2(x)|_{x=1}x = x \tag{4.14}$$

Direct calculations of the right-hand side of (4.12) imply that

$$f_1(x)f_1(1-x) \equiv 1, \quad f_1(0) = f_1(1/2) = f_1(1) = 1 \tag{4.15}$$

Let us show that function $f_1(x)$ is strictly convex on $x \in (0, 1/2)$. To prove this, we calculate $f'(x)$ and $f''(x)$. After direct calculations we get

$$f_1'(x) = f_1(x)g(x), \text{ where } g(x) = \frac{-3}{(1+x)(2-x)} + \frac{7}{(3+x)(4-x)} + \frac{5}{(2+x)(3-x)}, \tag{4.16}$$

and therefore

$$f_1'(0) = f_1'(1) = -1/12, \quad f'(1/2) = 4/105. \tag{4.17}$$

Changing variables by formulas $x = y + 1/2$ in g, defined in (4.16), we get that

$$g(y+1/2) = \frac{-3}{(\frac{9}{4}-y^2)} + \frac{7}{(\frac{49}{4}-y^2)} + \frac{5}{(\frac{25}{4}-y^2)}$$

therefore, for $x \in (0,1)$ or, equivalently, for $y \in (-1/2, 1/2)$

$$\partial_x g(x) = \partial_y g(y+1/2) = 2y\frac{\partial^2}{\partial y^2}g(y+1/2) \tag{4.18}$$

where

$$\frac{\partial^2 g(y+1/2)}{\partial y^2} = \frac{-3}{(\frac{9}{4}-y^2)^2} + \frac{7}{(\frac{49}{4}-y^2)^2} + \frac{25}{(\frac{25}{4}-y^2)^2} \le \left.\frac{-3}{(\frac{9}{4}-y^2)^2}\right|_{y^2=0}$$
$$+ \left.\left(\frac{7}{(\frac{49}{4}-y^2)^2} + \frac{25}{(\frac{25}{4}-y^2)^2}\right)\right|_{y^2=1/4} = -\frac{16}{27} + \frac{7}{144} + \frac{5}{36} < 0 \tag{4.19}$$

According to (4.16) $f_1''(x) = f_1(x)(g^2(x)+g'(x))$, therefore (4.18),(4.19) imply strict convexity of $f_1(x)$ on $(0,1/2)$, which means that function $f_1(x)$ has only one point of minimum x_-, and, from (4.16) we get

$$1 > f_1(x_-) > 1 + \min_{x\in(0,1/2)}(-x/12, 4(x-1/2)/105) = 151/153. \tag{4.20}$$

According to (4.20),(4.15) function $f_1(x)$ has a single point of maximum x_+ on $(1/2,1)$ satisfying $x_-x_+ = 1$, and

$$1 < f_1(x_+) < 153/151. \tag{4.21}$$

By virtue of (4.10),(4.12),(4.13),(4.14)

$$\max_{\substack{x,y\in[0,1]\times[0,1]\\ x+y\le 1}} I(x,y) = \max_{\substack{x,y\in[0,1]\times[0,1]\\ x+y\le 1}} f_1(x)f_1(y)f_2(x+y) \le \max_{\substack{x,y\in[0,1]\times[0,1]\\ x+y\le 1}} f_1(x)f_1(y)(x+y)$$

Now we only have to show that the right-hand side of this equality is equal to one. According to (4.15), $f_1(x)f_1(y)(x+y) = 1$ for $x+y=1$. Let us consider the case when $x+y<1$. Then

$$f_1(x)f_1(y)(x+y) < 1 \tag{4.22}$$

for $0 \le x,y \le 1/2$, as for such x,y $f_1 \le 1$. For $x_- \le y < 1/2 < x$ (4.22) is also true, since $f_1(y)f_1(x)(x+y) \le f_1(y)f_1(1-y)(x+y) < 1$ according to (4.15). Now let $0 \le y < x_-$ and $x > 1/2$. If $x \le x_+$, then $f_1(y)f_1(x)(x+y) \le f_1(y)f_1(x*)(x*+y)$, where $x* \ge x_+$ is such a point that $f_1(x) = f_1(x*)$, and the problem is reduced to the case $x \ge x_+$. Thus, now let $0 \le y < x_-$ and $x \ge x_+$. If $f_1(y)f_1(x) \le 1$, then (4.22) is true (since $x+y<1$). For $f_1(y)f_1(x) > 1$ there exists single $y* \in (y, x_-)$ such that $f_1(y*)f_1(x) = 1$. According to (4.17)

$$f_1(y) < f_1(0) - y\frac{f_1(0)-f_1(y*)}{y*} = f_1(y*) + (y*-y)\frac{f_1(0)-f_1(y*)}{y*} < f_1(y*) + \frac{y*-y}{12}$$

and we get by virtue of (4.15),(4.21) that

$$f_1(y)f_1(x)(x+y) < \left(f_1(y*) + \frac{y*-y}{12}\right) f_1(x)(x+y)$$

$$< \left(1 + \frac{f_1(x)}{12}(y*-y)\right)(1-(y*-y)) = 1 - \left(1 - \frac{f_1(x)}{12}\right)(y*-y) - \frac{f_1(x)}{12}(y*-y)^2 < 1.$$

This completes the proof of (4.9). □

5. Positiveness of $J_1(ii)$

Let $(a,b)\times(c,d)$ be an open rectangle. We denote by $J_1(\{(a,b)\times(c,d)\})$ the part of sum J_1, which contains $a_{k,m}$ with $(k,m)\in(a,b)\times(b,c)$. The positiveness of $J_1(\{(4p,+\infty)\times(0,p)\})$ and $J_1(\{(0,p)\times(4p,+\infty)\})$ was proved in [8] and does not present difficulties. In this section we will show that the positive summands from the square $J_1(\{(p,2p)\times(p,2p)\})$ and triangles $J_1(\{(2p,2p),(3p,p),(3p,2p)\})$ $J_1(\{(2p,2p),(p,3p),(2p,3p)\})$ compensate for the rest of the negative summands in $J_1(ii)$.

Lemma 5.1. *The following inequalities hold:*

$$0,09J_1(\{(p,2p)\times(p,2p)\})+\sum_{\alpha=2}^{+\infty}J_1(\{((2\alpha+1)p,(2\alpha+2)p)\times(p,2p)\})>0, \quad (5.1)$$

$$0,09J_1(\{(p,2p)\times(p,2p)\})+\sum_{\alpha=2}^{+\infty}J_1(\{(p,2p)\times((2\alpha+1)p,(2\alpha+2)p)\})>0. \quad (5.2)$$

Proof. By virtue of symmetry, it is enough to prove (5.1) only. Let us perform the change of variables $k=p+px, m=py$ in $J_1(\{(p,2p)\times(p,2p)\})$, and $k=(2\alpha+2)p+px, m=py,\ x\in(0,1), y\in(1,2)$ in $J_1(\{((2\alpha+1)p,(2\alpha+2)p)\times(p,2p)\})>0$, and consider the ratio of the summands corresponding to the equal values of x and y:

$$\begin{aligned}F_\alpha(x,y)&:=\left|\frac{A((2\alpha+1)p+px)A(py)A((2\alpha+1)p+px+py)}{A(p+px)A(py)A(p+px+py)}\right|\\&=\left|\frac{\hat D(2\alpha+1+x)\hat D(2\alpha+1+x+y)}{\hat D(1+x)\hat D(1+x+y)}\right|\cdot e^{-Kt},\end{aligned} \quad (5.3)$$

where $K=8\alpha(\alpha+1)p^2+4\alpha p^2(2x+y)>0$ and

$$\hat D(x)=\frac{x}{(x^2-4)(x^2-16)}. \quad (5.4)$$

Since $K>0$, it is enough to prove (5.3) for $t=0$. According to (5.4)

$$\begin{aligned}\left|\frac{\hat D(2\alpha+1+x)}{\hat D(1+x)}\right|&=\frac{(2\alpha+1+x)((1+x)^2-4)(((1+x)^2-16))}{((2\alpha+1+x)^2-4)((2\alpha+1+x)^2-16)(1+x)}\\&=\frac{2\alpha+1+x}{2\alpha-1+x}\cdot\frac{3+x}{1+x}\cdot\frac{3-x}{2\alpha-3+x}\cdot\frac{(1-x)(5+x)}{(2\alpha+3+x)(2\alpha+5+x)}\end{aligned} \quad (5.5)$$

Obviously, all the multipliers in the right hand side of (5.5), apart from the last, decrease monotonously. Let us prove that the last multiplier

$$g(x):=\frac{(1-x)(5+x)}{(2\alpha+3+x)(2\alpha+5+x)}$$

also decreases monotonously.

Indeed, for $x\in(0,1)$

$$\begin{aligned}g'(x)&=g(x)\left[\frac{2\alpha}{(2\alpha+5+x)(5+x)}-\frac{2\alpha+4}{(2\alpha+3+x)(1-x)}\right]\\&\geq-\frac{4\alpha g(x)(x^2+(2\alpha+6)x+4\alpha+11)}{(2\alpha+5+x)(2\alpha+3+x)(5+x)(1-x)}\end{aligned}$$

Since the roots of the square trinomial in the numerator of the right-hand side of the inequality are negative, then the trinomial itself is positive for $x \in (0,1)$, and $g'(x) < 0$. Therefore, for $x = 0$ in the right-hand side of (5.5) we get the estimate

$$\left|\frac{\hat{D}(2\alpha+1+x)}{\hat{D}(1+x)}\right| < \frac{45(2\alpha+1)}{(2\alpha-1)(2\alpha+3)(2\alpha-3)(2\alpha+5)}. \tag{5.6}$$

Similarly to (5.5) we get, that

$$\left|\frac{\hat{D}(2\alpha+1+z)}{\hat{D}(1+z)}\right| = \left[\frac{2\alpha+1+z}{2\alpha+3+z}\cdot\frac{5+z}{2\alpha+5+z}\cdot\frac{z-1}{1+z}\cdot\frac{3+z}{2\alpha-1+z}\right]\frac{3-z}{2\alpha-3+z}, \tag{5.7}$$

where $z = x+y, z \in (1,3)$.

Let us first study function (5.7) for $\alpha = 2$. It has got the following form:

$$f_1(z) := \frac{(5+z)^2}{(1+z)^2}\cdot\frac{z-1}{7+z}\cdot\frac{3-z}{9+z}. \tag{5.8}$$

On the interval $z \in (1;3)$ function $f_1(z)$ is positive, and turns into zero at its ends. The derivative of the function $f_1(z)$ is equal to

$$f_1'(z) = f_1(z)\left(\frac{2}{5+z} - \frac{2}{1+z} + \frac{1}{z-1} - \frac{1}{3-z} - \frac{1}{7+z} - \frac{1}{9+z}\right) = f_1(z)g(z) \tag{5.9}$$

where the last equality is the definition of function $g(z)$.

Let us calculate the derivative of $g(z)$:

$$g'(z) = g_1(z) + g_2(z),$$

where

$$g_1(z) = -\frac{2}{(5+z)^2} - \frac{1}{(z-1)^2}, \tag{5.10}$$

$$g_2(z) = \frac{2}{(1+z)^2} - \frac{1}{(3-z)^2} + \frac{1}{(7+z)^2} + \frac{1}{(9+z)^2}. \tag{5.11}$$

Obviously, $g_1(z)$ is increasing while $g_2(z)$ is decreasing for $z \in (1,3)$, therefore

$$g'(z) < g_1(3) + g_2(1) < 0.$$

Thus, function $g(z)$ is monotonously decreasing, taking values from $(-\infty,+\infty)$, and turning into zero in one point exactly. Therefore on the interval $(1,3)$ function $f_1(z)$ has got exactly one extremal point, which is obviously the point of maximum.

Let us consider function $f_1(z)$ on the interval $(1.6;\ 1.8)$. Since $0.275 < g(1.6) < 0.276$, and $-0.212 < g(1.8) < -0.211$, then the maximum of function $f_1(z)$ belongs to this interval. At the same time $g'(z) < g_1(1.8) + g_2(1.6) < -1.798$. Since $0 \le g^2(z) < 0.08$ on the interval $(1.6; 1.8)$, then the second derivative of the function $f_1(z)$, which, according to (5.9), equals

$$f_1''(z) = f_1(z)\left(g^2(z) + g'(z)\right),$$

is negative, and $f_1(z)$ is convex upwards on $(1.6; 1.8)$.

Let us consider the tangent lines to the graph of $f_1(z)$ at $z = 1.6$ and $z = 1.8$. Their equations have the form $h_1(z) = f_1'(1.6)(z-1.6) + f_1(1.6)$ and $h_2(z) =$

$f_1'(1.8)(z-1.8)+f_1(1.8)$ correspondingly. The graph of the function $f_1(z)$ lies below the point of intersection of these tangent lines, therefore,

$$f_1(z) < \frac{f_1'(1.6)(f_1(1.8)-1.8\cdot f_1'(1.8)) - f_1'(1.8)(f_1(1.6)-1.6\cdot f_1'(1.6))}{f_1'(1.6)-f_1'(1.8)} < 0.061. \tag{5.12}$$

Function (5.8) for $\alpha=3$ is studied similarly. It has the following form:

$$f_2(z) := \frac{(7+z)(z-1)(3-z)}{(9+z)(1+z)(11+z)}. \tag{5.13}$$

For $z\in(1;3)$ function $f_2(z)$ is positive and turns zero at the ends of the interval. The derivative of function $f_2(z)$ equals

$$f_2'(z) = f_2(z)\left(\frac{1}{7+z}+\frac{1}{z-1}-\frac{1}{3-z}-\frac{1}{9+z}-\frac{1}{1+z}-\frac{1}{11+z}\right) = f_1(z)q(z) \tag{5.14}$$

where the last equality is the definition of function $q(z)$.

Let us find the derivative of function $q(z)$:

$$q'(z) = q_1(z)+q_2(z),$$

where

$$q_1(z) = -\frac{1}{(7+z)^2}-\frac{1}{(z-1)^2}, \tag{5.15}$$

$$q_2(z) = -\frac{1}{(3-z)^2}+\frac{1}{(1+z)^2}+\frac{1}{(9+z)^2}+\frac{1}{(11+z)^2}. \tag{5.16}$$

Obviously, $q_1(z)$ increases and $q_2(z)$ decreases for $z\in(1,3)$, therefore

$$q'(z) < q_1(3)+q_2(1) < 0.$$

Thus, function $q(z)$ decreases monotonously, taking values in $(-\infty;+\infty)$, and turning into zero in exactly one point. Therefore, on the interval $(1,3)$ function $f_2(z)$ has got exactly one extremal point, which is obviously the point of maximum.

Let us consider function $f_2(z)$ on the interval $(1.7;\,1.9)$. Since $0.231 < q(1.7) < 0.232$, and $-0.2 < q(1.9) < -0.19$, then the maximum of function $f_2(z)$ belongs to this interval. At the same time $q'(z) < q_1(1.9)+q_2(1.7) < -1.687$. Therefore, the second derivative of function $f_2(z)$, which according to (5.14) equals

$$f_2''(z) = f_2(z)\left(q^2(z)+q'(z)\right),$$

is negative, and $f_2(z)$ is convex upwards on $(1.7;1.9)$.

Let us consider tangent lines to the graph of function $f_2(z)$ at points $z=1.7$ and $z=1.9$. Their equations have got the form $h_1(z)=f_2'(1.7)(z-1.7)+f_2(1.7)$ and $h_2(z)=f_2'(1.9)(z-1.9)+f_2(1.9)$ correspondingly. The graph of function $f_2(z)$ lies below the point of intersection of these two tangent lines, therefore

$$f_2(z) < \frac{f_2'(1.7)(f_2(1.9)-1.9\cdot f_2'(1.9)) - f_2'(1.9)(f_2(1.7)-1.7\cdot f_2'(1.7))}{f_2'(1.7)-f_2'(1.9)} < 0.0221. \tag{5.17}$$

As follows from estimates (5.6)(5.12)(5.17), for $\alpha=2$

$$F_\alpha(x,y) < 0.0714, \tag{5.18}$$

and for $\alpha=3$

$$F_\alpha(x,y) < 0.005. \tag{5.19}$$

Obviously, every multiplier in (5.8) increases, apart from the last, which decreases. Therefore, setting $z = 3$ in the multiplier within the square brackets in (5.8), and $z = 1$ in the rest multiplier, we get the estimate

$$\left| \frac{\hat{D}(2\alpha+1+z)}{\hat{D}(1+z)} \right| < \frac{6(\alpha+2)}{(\alpha-1)(\alpha+1)(\alpha+3)(\alpha+4)}. \tag{5.20}$$

Therefore,

$$F_\alpha(x,y) < \frac{270(\alpha+2)(2\alpha+1)}{(\alpha-1)(\alpha+1)(\alpha+3)(\alpha+4)(2\alpha-3)(2\alpha-1)(2\alpha+3)(2\alpha+5)} := I(\alpha) \tag{5.21}$$

where the last equation is the definition of function $I(\alpha)$.

Our goal is to find the upper bound for expression

$$\sum_{\alpha=2}^{\infty} I(\alpha) = I(4) + \sum_{\alpha=5}^{\infty} I(\alpha) \tag{5.22}$$

It is easy to see, that

$$I(4) = \frac{243}{112112} < 0,00217 \tag{5.23}$$

and

$$\begin{aligned} \sum_{\alpha=5}^{\infty} I(\alpha) &\le 270 \sum_{\alpha=5}^{\infty} \frac{1}{(\alpha^2-1)(\alpha+4)(2\alpha-3)(2\alpha-1)(2\alpha+5)} \\ &\le 270 \int_{\alpha=4}^{\infty} \frac{d\alpha}{\alpha^2-1)(\alpha+5/2)(2\alpha-3)(2\alpha-1)(2\alpha+4)} \end{aligned} \tag{5.24}$$

Let us note that for $\alpha \ge 4$ inequality $(\alpha^2-1)(\alpha+5/2) = \alpha^3 + 2,5\alpha^2 - \alpha - 2,5 \ge \alpha^3$ is true, therefore both roots of the quadratic trinomial $\alpha^3 + 2,5\alpha^2 - \alpha - 2,5$ are less than 4. In the same way, $(x-3)(x-1)(x+8) = x^3 + 4x^2 - 29x + 24 > x^3$ for $x \ge 8$, because both roots of trinomial $4x^2 - 29x + 24$ are less than eight. Therefore the right hand side of inequality (5.24) allows estimate

$$\begin{aligned} \sum_{\alpha=5}^{\infty} I(\alpha) &\le 270 \int_{\alpha=4}^{\infty} \frac{d\alpha}{(\alpha^2-1)(\alpha+4)(2\alpha-3)(2\alpha-1)(2\alpha+5)} \\ &\le \frac{270}{8} \int_{\alpha=4}^{\infty} \frac{d\alpha}{\alpha^6} = \frac{27}{4^6} \le 0,007 \end{aligned} \tag{5.25}$$

The estimate

$$\frac{\sum_{\alpha=2}^{+\infty} J_1\{((2\alpha+1)p, (2\alpha+2)p) \times (p, 2p)\}}{J_1\{(p, 2p)} < 0,09 \tag{5.26}$$

that follows from (5.23)-(5.25) completes the proof of Lemma 5.1. □

The next lemma completes this section:

Lemma 5.2. *The following inequalities are true:*

$$\begin{aligned} &0.22 J_1(\{(2p,2p),(3p,2p),(3p,p)\}) \\ &\quad + \sum_{\alpha=2}^{+\infty} J_1(\{(2\alpha p, 2p), ((2\alpha+1)p, 2p), ((2\alpha+1)p, p)\}) > 0, \end{aligned} \tag{5.27}$$

$$0.22J_1(\{(2p,2p),(2p,3p),(p,3p)\}) + \sum_{\alpha=2}^{+\infty} J_1(\{(2p,2\alpha p),((2p,2\alpha+1)p),(p,(2\alpha+1)p)\}) > 0, \tag{5.28}$$

Proof. For reasons of symmetry, it is enough to prove only (5.27). Let us perform the change of variables $k = 3p - px, m = 2p - py$ in $J_1(\{(2p,2p),(3p,2p),(3p,p)\})$, and the change of variables $k = (2\alpha+1)p - px, m = 2p - py$ in $J_1(\{(2\alpha p,2p), ((2\alpha+1)p,2p),((2\alpha+1)p,p)\})$, $x, y \in (0,1), x + y < 1$. Consider now the ratio of summands corresponding to the same values of x and y:

$$\left|\frac{A((2\alpha+1)p-px)A(2p-py)A((2\alpha+3)p-px-py)}{A(3p-px)A(2p-py)A(5p-px-py)}\right| = \left|\frac{\hat{D}(2\alpha+1-x)\hat{D}(2\alpha+3-x-y)}{\hat{D}(3-x)\hat{D}(5-x-y)}\right| \cdot e^{-Kt}, \tag{5.29}$$

where $\hat{D}(x)$ is the function defined in (5.4), $K = 4p^2(\alpha-1)(2\alpha+6-2x-y > 0$ for $\alpha \geq 2$, $x, y \in (0,1)$. It is sufficient to find the upper bound of (5.29) for $t = 0$. Let

$$F_1(x,\alpha) := \left|\frac{\hat{D}(2\alpha+1-x)}{\hat{D}(3-x)}\right|, \qquad F_2(z,\alpha) := \left|\frac{\hat{D}(2\alpha+3-z)}{\hat{D}(5-z)}\right| \tag{5.30}$$

According to (5.4), (5.33)

$$F_1(x,\alpha) = \frac{2\alpha+1-x}{2\alpha-1-x}\frac{5-x}{2\alpha+3-x}\frac{7-x}{2\alpha-3-x}\frac{1-x}{2\alpha+5-x}\frac{1+x}{3-x},\ x \in (0,1) \tag{5.31}$$

$$F_2(z,\alpha) = \frac{2\alpha+3-z}{2\alpha+5-z}\frac{3-z}{2\alpha-1-z}\frac{7-z}{2\alpha+1-z}\frac{1-z}{5-z}\frac{9-z}{2\alpha+7-x},\ z \in (0,1). \tag{5.32}$$

For $\alpha = 2$ function (5.31)(5.32) has the following form:

$$F_1(x,2) = \frac{(5-x)^2(1+x)}{(3-x)^2(9-x)}, \tag{5.33}$$

$$F_2(z,2) = \frac{(7-z)^2(1-z)}{(5-z)^2(11-z)}. \tag{5.34}$$

Since every multiplier in (5.33) increases,

$$F_1(x,2) < F_1(1,2) = 1 \qquad x \in (0,1). \tag{5.35}$$

Next,

$$F_2'(z,2) = \frac{7-z}{5-z} \cdot \frac{4}{(5-z)^2} \cdot \frac{1-z}{11-z} - \left(\frac{7-z}{5-z}\right)^2 \frac{10}{(11-z)^2} = -\frac{6F_2(z,2)(z^2-12z+51)}{(1-z)(5-z)(7-z)(11-z)}$$

Since the discriminant of the quadratic trinomial in the numerator of the fraction in the right-hand side of the equation is negative, $F_2(z,2)$ decreases, and, taking into account of (5.35) we finally get, that

$$F_1(x,2)F_2(z,2) < F_1(1,2)F_2(0,2) = \frac{49}{275} \qquad x, z \in (0,1). \tag{5.36}$$

For $\alpha \geq 3$ every multiplier in (5.32) decreases monotonously, therefore

$$F_2(z,\alpha) < F_2(0,\alpha) = \frac{189(2\alpha+3)}{5(2\alpha-1)(2\alpha+1)(2\alpha+5)(2\alpha+7)}. \tag{5.37}$$

Since $1 + x < 3 - x$ for $x \in (0, 1)$, then, substituting $(3 - x)$ for $(1 + x)$ in the numerator of (5.31), we get, that

$$F_1(x, \alpha) < \frac{(2\alpha + 1 - x)}{(2\alpha + 3 - x)} \frac{(5 - x)}{(2\alpha - 1 - x)} \frac{(7 - x)}{(2\alpha + 5 - x)} \frac{(1 - x)}{(2\alpha - 3 - x)}. \tag{5.38}$$

Every multiplier in (5.38) decreases monotonously, therefore

$$F_1(x, \alpha) < \frac{35(2\alpha + 1)}{(2\alpha - 1)(2\alpha + 3)(2\alpha - 3)(2\alpha + 5)}. \tag{5.39}$$

Thus, (5.37) and (5.39) imply the estimate

$$F_1(x, \alpha)F_2(z, \alpha) < \frac{1323}{(2\alpha - 1)^2(2\alpha + 5)^2(2\alpha + 7)(2\alpha - 3)} := \hat{I}(\alpha), \tag{5.40}$$

where the last equality in (5.40) is the definition of $\hat{I}(\alpha)$ for $\alpha \geq 3$. According to (5.36), let us assume $\hat{I}(2) = \frac{49}{275}$.

Similarly to (5.23) and (5.24) according to (5.36), (5.40) we get

$$\hat{I}(2) = \frac{49}{275} < 0.18, \quad \hat{I}(3) = \frac{7}{605} < 0.012 \tag{5.41}$$

and

$$\sum_{\alpha=4}^{\infty} I(\alpha) \leq \int_3^{\infty} \hat{I}(\alpha) d\alpha = \frac{1323}{2} \int_6^{\infty} \frac{dx}{(x - 1^2)(x + 5)^2(x + 7)(x - 3)} \tag{5.42}$$

Since inequalities $(x - 1)(x + 5) = x^2 + 4x - 5 > x^2$ and $(x + 7)(x - 3) > x^2$ are true for $x \geq 6$, then

$$\frac{1323}{2} \int_6^{\infty} \frac{dx}{(x - 1^2)(x + 5)^2(x + 7)(x - 3)} < \frac{1323}{2} \int_6^{\infty} \frac{dx}{x^6} = \frac{161}{8640} < 0.019 \tag{5.43}$$

Relations (5.41) - (5.43) imply the estimate

$$\frac{\sum_{\alpha=2}^{\infty} J_1(\{(2\alpha p, 2p), ((2\alpha + 1)p, 2p), ((2\alpha + 1)p, p)\})}{J_1(\{(2p, 2p), (3p, 2p), (3p, p)\})} < 0.211, \tag{5.44}$$

which completes the proof of Lemma 5.2. □

6. Positiveness of $J_1(iii)$

In this section we prove the positiveness of the part of sum J_1 from the region $iii)$ and its neighborhood.

6.1. **Positiveness of the main part of $J_1(iii)$.** Let us establish the positiveness of the part of sum $J_1(iii)$ without summands, corresponding to $k, n \in \{(4p, 4p), (4p, 2p), (6p, 2p)\} \cup \{(4p, 4p), (2p, 4p), (2p, 6p)\}$.

The following statement is true:

Lemma 6.1. *Let $\alpha \in \mathbb{N}$, $2 \leq \alpha \leq 100$. Then the following inequalities hold:*

$$\begin{aligned} &J_1(\{(2p\alpha, 4p), (2p(\alpha + 1), 4p), (2p(\alpha + 1), 2p)\}) \\ &+ J_1(\{(2p(\alpha + 1), 2p), (2p(\alpha + 2), 2p), (2p(\alpha + 1), 4p)\}) > 0. \end{aligned} \tag{6.1}$$

$$\begin{aligned} &J_1(\{(4p, 2p\alpha), (4p, 2p(\alpha + 1)), (2p, 2p(\alpha + 1))\}) \\ &+ J_1(\{(2p, 2p(\alpha + 1)), (2p, 2p(\alpha + 2)), (4p, 2p(\alpha + 1))\}) > 0. \end{aligned} \tag{6.2}$$

Proof. Since the change of variables $(k,m) \to (m,k)$ turns the sum in (6.1) into the sum in (6.2), it is enough to prove inequality (6.1) only.

According to Lemma 3.2 the summands in $J_1(\{(2p\alpha, 4p), (2p(\alpha+1), 4p), (2p(\alpha+1), 2p)\})$ are positive, and the summands in $J_1(\{(2p(\alpha+1), 2p), (2p(\alpha+2), 2p), (2p(\alpha+1), 4p)\})$ are negative. Just as in the proof of Lemmas 4.2 4.3, let us consider the ratio of absolute values of negative and positive summands and show that this ratio does not exceed one.

Let us perform the change of variables $k = 2(\alpha+1)p - l, m = 4p - q$ in the sum $J_1(\{(2p\alpha, 4p), (2p(\alpha+1), 4p), (2p(\alpha+1), 2p)\})$ and the change $k+m = 2(\alpha+3)p-l$, $m = 2p+q$ in the sum $J_1(\{(2p(\alpha+1), 2p), (2p(\alpha+2), 2p), (2p(\alpha+1), 4p)\})$, where the variables in both sums change within bounds

$$0 < l < 2p, \quad 0 < q < 2p, \quad l + q < 2p.$$

Considering the ratio of the summands, corresponding to equal values of l and q and taking into account (4.5) we get

$$\begin{aligned} &\left|\frac{A(2(\alpha+2)p - l - q)A(2p+q)A(2(\alpha+3)p - l)}{A(2(\alpha+1)p - l)A(4p-q)A(2(\alpha+3)p - l - q)}\right| \\ &= \left|\frac{D(2(\alpha+2)p - l - q)D(2p+q)D(2(\alpha+3)p - l)}{D(2(\alpha+1)p - l)D(4p-q)D(2(\alpha+3)p - l - q)}\right| \cdot e^{-Kt}, \end{aligned} \tag{6.3}$$

where coefficient K in the exponent is obviously equal to

$$K = (2(\alpha+2)p-l-q)^2 - (2(\alpha+3)p-l-q)^2 + (2p+q)^2 - (4p-q)^2 + (2(\alpha+3)p-l)^2$$
$$-(2(\alpha+1)p - l)^2 = 4p(2\alpha p - l + 4q) > 0.$$

So it is enough to show, that the right-hand side of (6.3) is less than one for $t = 0$ to prove Lemma 6.1 .

Substituting variables x, y, defined as $l = 2px, q = 2py$ for variables l, q, and taking into account (4.1),(4.11) we get the following equation:

$$\left|\frac{\tilde{D}(\alpha+2-x-y)\tilde{D}(1+y)\tilde{D}(\alpha+3-x)}{\tilde{D}(\alpha+1-x)\tilde{D}(2-y)\tilde{D}(\alpha+3-x-y)}\right| = f_1(x)f_2(y)f_3(x+y) \tag{6.4}$$

where $0 < x < 1,\ 0 < y < 1,\ 0 < x+y < 1$ and

$$f_1(x) = \frac{\tilde{D}(\alpha+3-x)}{\tilde{D}(\alpha+1-x)}, \quad f_2(y) = \frac{\tilde{D}(1+y)}{\tilde{D}(2-y)}, \quad f_3(x+y) = \frac{\tilde{D}(\alpha+2-x-y)}{\tilde{D}(\alpha+3-x-y)} \tag{6.5}$$

Let us find the upper bound for each of the functions. According to (4.11)

$$\begin{aligned} f_1(x) &= \frac{(\alpha+3-x)((\alpha+1-x)^2-4)((\alpha+1-x)^2-1)}{((\alpha+3-x)^2-4)((\alpha+3-x)^2-1)(\alpha+1-x)} \\ &= \frac{\alpha+3-x}{\alpha+5-x} \cdot \frac{\alpha+3-x}{\alpha+4-x} \cdot \frac{\alpha-x}{\alpha+1-x} \cdot \frac{\alpha-1-x}{\alpha+1-x} \end{aligned} \tag{6.6}$$

Each of the multipliers in the right-hand side of the last of the equalities is a decreasing function, therefore

$$\max_{0<x<1} f_1(x) = f_1(0) = \frac{(\alpha+3)^2\alpha(\alpha-1)}{(\alpha+1)^2(\alpha+5)(\alpha+4)} < 1 \qquad \forall \alpha \geq 2 \tag{6.7}$$

Next,

$$f_3(z) = \frac{(\alpha+2-z)((\alpha+3-z)^2-4)((\alpha+3-z)^2-1)}{((\alpha+2-z)^2-4)((\alpha+2-z)^2-1)(\alpha+3-z)} = \left(\frac{\alpha+2-z}{\alpha+3-z}\right)^2 \cdot \frac{\alpha+5-z}{\alpha-z} \tag{6.8}$$

and

$$f_3'(z) = f_3(z)\left[\frac{-2}{(\alpha+3-z)(\alpha+2-z)} + \frac{5}{(\alpha-z)(\alpha+5-z)}\right] = \frac{3f_3(z)(z^2-(2\alpha+5)z+\alpha^2+5\alpha+10)}{(\alpha+3-z)(\alpha+2-z)(\alpha-z)(\alpha+5-z)} \tag{6.9}$$

Since the discriminant of the quadratic trinomial in the numerator from the right-hand side of the last equality is negative, function $f_3(z)$ increases, and therefore

$$\max_{0<z<1} f_3(z) = f_3(1) = \frac{(\alpha+1)^2(\alpha+4)}{(\alpha+2)^2(\alpha-1)} \tag{6.10}$$

Finally,

$$f_2(y) = \frac{(1+y)(4-(2-y)^2))((2-y)^2-1)}{(4-(1+y)^2)((1+y)^2-1)(2-y)} = \frac{1+y}{2-y} \cdot \frac{3-y}{2+y} \cdot \frac{4-y}{3+y} \tag{6.11}$$

To estimate function $f_2(y)$, let us rewrite it as follows:

$$f_2(y) = 1 + \frac{2y^3-3y^2+y}{12+4y-3y^2-y^3}. \tag{6.12}$$

The numerator of the fraction in (6.12) reaches its maximum on $y \in (0,1)$ at $y = \frac{3-\sqrt{3}}{6}$. Since the roots of the cubic polynomial $h(y) = 12+4y-3y^2-y^3$ in the denominator of the fraction are equal to $2, -2, -3$, $h(y)$ has got just one extremal point (the point of maximum) on the interval $(0,1)$. Equalities $h(0) = h(1) = 12$ imply that h reaches its minimum on $[0,1]$ at the ends of the segment.

Therefore, the estimate

$$f_2(y) < 1 + \frac{1}{12} \cdot \left(2\left(\frac{3-\sqrt{3}}{6}\right)^3 - 3\left(\frac{3-\sqrt{3}}{6}\right)^2 + \frac{3-\sqrt{3}}{6}\right) < 1.017 \tag{6.13}$$

is true.

Relations (6.7), (6.10), (6.13) imply that

$$\max_{0<x<1, 0<y<1} f_1(x)f_2(y)f_3(x+y) < f_1(0)f_3(1) \cdot 1.017 = 1.021 \cdot \frac{(\alpha+3)^2\alpha}{(\alpha+2)^2(\alpha+5)} := 1.017 \cdot g(\alpha), \tag{6.14}$$

where the last equality is the definition of function $g(\alpha)$. Obviously,

$$g'(\alpha) = g(\alpha)\left[\frac{-2}{(\alpha+3)(\alpha+2)} + \frac{5}{\alpha(\alpha+5)}\right] = \frac{3g(\alpha)(\alpha^2+5\alpha+6)}{\alpha(\alpha+2)(\alpha+3)(\alpha+5)} > 0,\ \alpha > 0$$

so $g(\alpha) \le g(100)$ for $2 \le \alpha \le 100$, therefore,

$$\max_{0<x<1, 0<y<1} f_1(x)f_2(y)f_3(x+y) < 1, \tag{6.15}$$

and Lemma6.1 is proved. $\square$

The next lemma completes the proof of the positiveness of the considered part of the sum $J_1(iii)$:

Lemma 6.2. *The following inequality holds:*

$$\sum_{m=2p+1}^{4p-1}\left(B(p)B(m)B(p+m)+\sum_{k=200p+1}^{+\infty}A(k)A(m)A(k+m)\right)>0 \qquad (6.16)$$

Proof. For every $m\in[2p+1;4p-1]$ bound (6.16) follows from the inequality

$$B(p)B(p+m)>-\sum_{k=200p+1}^{+\infty}A(k)A(k+m). \qquad (6.17)$$

Let us first estimate the expression from the right-hand side of (6.17). According to the definition of $A(k)$ and Lemma4.1,

$$\begin{aligned}&\left|\sum_{k=200p+1}^{\infty}A(k)A(k+m)\right|\le\sum_{k=200p+1}^{\infty}D(k)D(k+m)\\ &<\sum_{k=200p+1}^{\infty}D^2(k)=\sum_{k=200p+1}^{\infty}\frac{k^2}{(k^2-4p^2)(k^2-16p^2)}\\ &\le\sum_{k=200p+1}^{\infty}\frac{1}{k^6}=\frac{1}{p^6}\sum_{px=200p+1}^{\infty}\frac{1}{x^6}\le\frac{1}{p^6}\int_{200}^{+\infty}\frac{dx}{x^6}=\frac{1}{5\cdot(200)^5p^6}.\end{aligned} \qquad (6.18)$$

Next, let us consider the function

$$B(k)=\frac{\sin\frac{\pi k}{2p}}{k-4p}\cdot\hat{B}(k),$$

where

$$\hat{B}(k)=\frac{k}{(k-2p)(k+2p)(k+4p)}.$$

Since $\sin x>\frac{2}{\pi}x$, $x\in(0,\frac{\pi}{2})$ and $-\sin x>-\frac{2}{\pi}x$, $x\in(-\frac{\pi}{2},0)$, we get

$$\frac{\sin\frac{\pi k}{2p}}{k-4p}=\frac{\sin\frac{\pi q}{2p}}{q}>1,\ \ k=4p+q,\ q\in(-p,p)$$

Therefore, for $m\in(2p,4p)$

$$B(p+m)=\hat{B}(p+m).$$

Since function $\hat{B}(p+m)$ decreases monotonously for $p+m>2p$, the following estimate is true:

$$B(p)B(p+m)\ge B(p)\hat{B}(5p)=\frac{1}{9\cdot 189p^6} \qquad (6.19)$$

Relations (6.18) and (6.19) complete the proof of (6.16)

$\square$

6.2. **Positiveness of** $J_1(\{(p,2p)\times(p,2p)\})+J_1(\{(4p,5p)\times(2p,3p)\})+J_1(\{(2p,3p)\times(4p,5p)\})$**.** In Lemma5.1 of section 5 we employed only $0.18J_1(\{(p;2p)\times(p;2p)\})$. In this section we will show that sum $J_1(\{(p;2p)\times(p;2p)\})$ compensates for negative summands in $J_1(\{(4p;5p)\times(2p;3p)\})$ and $J_1(\{(2p;3p)\times(4p;5p)\})$ as well.

Lemma 6.3. *The following estimates are true:*

$$0.28J_1(\{(p,2p)\times(p,2p)\})+J_1(\{(4p,5p)\times(2p,3p)\})>0; \tag{6.20}$$

$$0.28J_1(\{(p,2p)\times(p,2p)\})+J_1(\{(2p,3p)\times(4p,5p)\})>0; \tag{6.21}$$

Proof. Since change of variables $(k,m)\to(m,k)$ turns inequality (6.20) into (6.21), it is enough to prove only (6.20). Let us perform the change of variables $k=2p-px$, $m=2p-py$ in $J_1\{(p,2p)\times(p,2p)\}$ and $k=4p+px, m=2p+py$ in $J_1\{(4p,5p)\times(2p,4p)\}$, $x,y\in(0,1)$, and consider the ratio of the summands, corresponding to the equal values of x and y:

$$\begin{aligned}&\left|\frac{A(4p+px)A(2p+py)A(6p+px+py)}{A(2p-px)A(2p-py)A(4p-px-py)}\right|\\ =&\frac{\hat{D}(4+x)\hat{D}(2+y)\hat{D}(6+x+y)}{\hat{D}(2-x)\hat{D}(2-y)\hat{D}(4-x-y)}\cdot e^{-Kt},\end{aligned} \tag{6.22}$$

where $K=8p^2(8+8x+7y)>0$ and $\hat{D}(x)$ is defined in (5.4).

Let us find the lower bound for (6.22) for $t=0$.

For $t=0$ fraction (6.22) has got the following form:

$$F(x,y):=\frac{\hat{D}(4+x)}{\hat{D}(2-x)}\frac{\hat{D}(2+y)}{\hat{D}(2-y)}\frac{\hat{D}(6+x+y)}{\hat{D}(4-x-y)}=f_1(x)f_2(y)f_3(x+y), \tag{6.23}$$

where, according to (5.4),

$$\begin{aligned}f_1(x)=&\frac{\hat{D}(4+x)}{\hat{D}(2-x)}=\frac{(4+x)((2-x)^2-4)((2-x)^2-16)}{((4+x)^2-4)((4+x)^2-16)(2-x)}\\ =&\frac{4+x}{6+x}\frac{6-x}{8+x}\frac{4-x}{2-x},\end{aligned} \tag{6.24}$$

$$\begin{aligned}f_2(y)=&\frac{\hat{D}(2+y)}{\hat{D}(2-y)}=\frac{(2+y)((2-y)^2-4)((2-y)^2-16)}{((2+y)^2-4)((2+y)^2-16)(2-y)}\\ =&\left(\frac{2+y}{2-y}\right)^2\frac{4-y}{4+y}\frac{6-y}{6+y},\end{aligned} \tag{6.25}$$

$$\begin{aligned}f_3(z)=&\frac{\hat{D}(6+z)}{\hat{D}(4-z)}=\frac{(6+z)((4-z)^2-4)((4-z)^2-16)}{((6+z)^2-4)((6+z)^2-16)(4-z)}\\ &\frac{z(2-z)(6-z)(6+z)(8-z)}{(2+z)(4-z)(4+z)(8+z)(10+z)}, z\in(0,2).\end{aligned} \tag{6.26}$$

Let us estimate each of the functions (6.24)-(6.26).

Let us find the derivative of function (6.24):

$$\begin{aligned}f_1'(x)=&2f_1(x)\left(\frac{32+4x+2x^2}{(4+x)(6+x)(4-x)(2-x)}-\frac{7}{(6-x)(8+x)}\right)\\ =&f_1(x)\left(h_1(x)-h_2(x)\right),\end{aligned}$$

where the last equality is the definition of $h_1(x)$, $h_2(x)$.

The numerator of $h_1(x)$ increases for $x \in (0,1)$ and the denominator, equal to $(16-x^2)(12-4x-x^2)$, decreases, therefore $h_1(x)$ increases for $x \in (0,1)$.

The denominator of $h_2(x)$ is equal to $(48-2x-x^2)$ and decreases for $x \in (0,1)$, therefore function $h_2(x)$ itself increases.

Thus,

$$f_1'(x) > 2f_1(x)\left(h_1(0) - h_2(1)\right) = \frac{1}{90} > 0,$$

and function $f_1(x)$ increases monotonously for $x \in (0,1)$.

Next, let us find the derivative of $f_2(y)$:

$$\begin{aligned} f_2'(y) &= 4f_2(y)\left(\frac{2}{4-y^2} - \frac{2}{16-y^2} - \frac{3}{36-y^2}\right) \\ &= 12f_2(y)\frac{244+12y^2-y^4}{(4-y^2)(16-y^2)(36-y^2)} > 0. \end{aligned}$$

Functions $f_1(x)$ and $f_2(y)$ are increasing, therefore

$$f_1(x)f_2(y) < f_1(1)f_2(1) = \frac{25}{21}\cdot\frac{27}{7} = \frac{225}{49}. \tag{6.27}$$

Then let us consider function $f_3(z)$ on the interval $(0,2)$. Function $f_3(z)$ turns into zero at the ends of the interval and is positive within. Its derivative equals

$$\begin{aligned} f_3'(z) = f_3(z)\Bigg(&-\frac{1}{2-z} + \frac{1}{6+z} - \frac{1}{8-z} - \frac{1}{6-z} + \frac{1}{z} \\ &-\frac{1}{4+z} + \frac{1}{4-z} - \frac{1}{8+z} - \frac{1}{10+z} - \frac{1}{2+z}\Bigg) =: f_3(z)g(z), \end{aligned} \tag{6.28}$$

where the last equality is the definition of $g(z)$.

The derivative of function $g(z)$ is equal to

$$g'(z) = g_1(z) + g_2(z),$$

where

$$\begin{aligned} g_1(z) &= \left(-\frac{1}{(2-z)^2} - \frac{1}{(6-z)^2} - \frac{1}{(8-z)^2} + \frac{1}{(2+z)^2} + \frac{1}{(4+z)^2} + \frac{1}{(8+z)^2}\right), \\ g_2(z) &= \left(-\frac{1}{z^2} - \frac{1}{(6+z)^2} + \frac{1}{(4-z)^2} + \frac{1}{(10+z)^2}\right). \end{aligned}$$

Obviously, function $g_1(z)$ decreases, and $g_2(z)$ increases, therefore,

$$g'(z) < g_1(0) + g_2(1) < -0.864, z \in (0,1], \tag{6.29}$$

$$g'(z) < g_1(1) + g_2(2) < -0.944, z \in (1,2). \tag{6.30}$$

Thus, $g'(z) < -0.864, z \in (0,2)$, therefore, function $g(z)$ decreases for $z \in (0,2)$, taking values from $(-\infty;+\infty)$, and turns into zero at exactly one point. Thus, function $f_3(z)$ has got a single maximum point on $(0,2)$.

Function $g(z)$ is monotonous, therefore, since $0.622 < g(1/2) < 0.623$, and $-0.603 < g(1) < -0.602.$, function $f_3(z)$ increases at $(0,1/2)$ and decreases at $(1,2)$. Let us consider function $f_3(z)$ at $(1/2;1)$.

According to (6.28),

$$f_3''(z) = f_3(z)\left(g^2(z) + g'(z)\right).$$

Since $0 \leq g^2(z) < 0.389$ for $z \in (1/2;1)$, by estimates (6.29),(6.30) the second derivative of $f_3(z)$ is negative on this interval. Therefore, $f_3(z)$ is convex upwards an its graph lies below the tangent lines at every point $z \in (1/2, 1)$.

Let us consider the tangent line to the graph of $f_3(z)$ at $z = 0.7$. It is given by equation $h(z) = f_3'(0.7)(z-0.7)+f_3(0.7)$, where $f_3'(0.7) < -3.37 \cdot 10^{-4}$. Therefore,

$$f_3(z) \leq h(z) < h(0.5) < 0.0606. \tag{6.31}$$

Inequalities (6.27) and (6.31) imply the estimate

$$F(x,y) < \frac{225}{49} \cdot 0.0606 < 0.28,$$

which completes the proof of (6.20). □

6.3. **Positiveness of** $J_1(\{(5p, 2p), (6p, 2p), (5p, 3p)\}) + J_1(\{(3p, 2p), (2p, 2p), (3p, p)\})$ **and** $J_1(\{(2p, 5p), (2p, 6p), (3p, 5p)\}) + J_1(\{(2p, 3p), (2p, 2p), (p, 3p)\})$**.** In this subsection we will show, that the remaining parts of the sums $J_1(\{(3p, 2p), (2p, 2p), (3p, p)\})$ and $J_1(\{(2p, 3p), (2p, 2p), (p, 3p)\})$, considered in Lemma5.2 compensate for the negative summands from $J_1(\{(5p, 2p), (6p, 2p), (5p, 3p)\})$ and $J_1(\{(2p, 5p), (2p, 6p), (3p, 5p)\})$ correspondingly.

Lemma 6.4. *The following inequalities hold:*

$$\frac{7}{33} J_1(\{(3p, 2p), (2p, 2p), (3p, p)\}) + J_1(\{(5p, 2p), (6p, 2p), (5p, 3p)\}) > 0; \tag{6.32}$$

$$\frac{7}{33} J_1(\{(2p, 3p), (2p, 2p), (p, 3p)\}) + J_1(\{(2p, 5p), (2p, 6p), (3p, 5p)\}) > 0. \tag{6.33}$$

Proof. Because of the symmetry, it is sufficient to prove only inequality (6.32).

Let us perform the change of variables $k = 3p-px, m = 2p-py$ in $J_1(\{(3p, 2p), (2p, 2p), (3p, p)\})$, and the change $k = 5p+px, m = 2p+py$ in $J_1(\{(5p, 2p), (6p, 2p), (5p, 3p)\})$, $x, y \in (0, 1), x + y < 1$, and consider the ratio of summands with equal values of x and y:

$$\begin{aligned} &\left|\frac{A(5p+px)A(2p+py)A(7p+px+py)}{A(3p-px)A(2p-py)A(5p-px-py)}\right| \\ &= \left|\frac{\hat{D}(5+x)\hat{D}(2+y)\hat{D}(7+x+y)}{\hat{D}(3-x)\hat{D}(2-y)\hat{D}(5-x-y)}\right| \cdot e^{-Kt}, \end{aligned} \tag{6.34}$$

where $K = 8p^2(5+5x+4y) > 0$ for $x, y \in (0, 1)$, and function $\hat{D}(x)$ is defined in (5.4).

For $t = 0$ the right-hand side of (6.34) can be written as

$$F(x,y) := f_1(x)f_2(y)f_3(x+y), \tag{6.35}$$

where, according to (5.4),

$$\begin{aligned} f_1(x) = \frac{\hat{D}(5+x)}{\hat{D}(3-x)} &= \frac{(5+x)((3-x)^2-4)(16-(3-x)^2)}{((5+x)^2-4)((5+x)^2-16)(3-x)} \\ &= \frac{5+x}{3+x} \cdot \frac{1-x}{3-x} \cdot \frac{5-x}{7+x} \cdot \frac{7-x}{9+x} \end{aligned} \tag{6.36}$$

$$
\begin{aligned}
f_2(x) = \frac{\hat{D}(2+y)}{\hat{D}(2-y)} &= \frac{(2+y)(4-(2-y)^2)(16-(2-y)^2)}{((2+y)^2-4)(16-(2+y)^2)(2-y)} \\
&= \frac{2+y}{4+y}\cdot\frac{2+y}{6+y}\cdot\frac{4-y}{2-y}\cdot\frac{6-y}{2-y}
\end{aligned} \tag{6.37}
$$

$$
\begin{aligned}
f_3(z) = \frac{\hat{D}(7+z)}{\hat{D}(5-z)} &= \frac{(7+z)((5-z)^2-4)((5-z)^2-16)}{((7+z)^2-4)((7+z)^2-16)(5-z)} \\
&= \frac{7+z}{5+z}\cdot\frac{3-z}{9+z}\cdot\frac{7-z}{3+z}\cdot\frac{1-z}{5-z}\cdot\frac{9-z}{11+z}
\end{aligned} \tag{6.38}
$$

Since each multiple in $f_1(x)$ and $f_3(z)$ decreases, and every multiple in $f_2(y)$ increases,

$$
F(x,y) < f_1(0)f_2(1)f_3(0) = \frac{7}{33}. \tag{6.39}
$$

The estimate (6.43) completes the proof of Lemma6.32. □

6.4. **Positiveness of** $J_1(\{(2p,p),(2p,2p),(3p,p)\}) + J_1(\{(4p,3p),(4p,4p),(5p,3p)\})$ **and** $J_1(\{(p,2p),(2p,2p),(p,3p)\}) + J_1(\{(3p,4p),(4p,4p),(3p,5p)\})$**.** This section completes the proof of the sum J_1 positiveness in region *iii*) and in its neighbourhood. Let us show that sums $J_1(\{(2p,p),(2p,2p),(3p,p)\})$ and $J_1(\{(p,2p),(2p,2p),(p,3p)\})$ compensate for negative summands from $J_1(\{(4p,3p),(4p,4p),(5p,3p)\})$ and $J_1(\{(3p,4p),(4p,4p),(3p,5p)\})$ correspondingly.

Lemma 6.5. *The following inequalities are true:*

$$
\frac{7}{33}J_1(\{(2p,p),(2p,2p),(3p,p)\}) + J_1(\{(4p,3p),(4p,4p),(5p,3p)\}) > 0; \tag{6.40}
$$

$$
\frac{7}{33}J_1(\{(p,2p),(2p,2p),(p,3p)\}) + J_1(\{(3p,4p),(4p,4p),(3p,5p)\}) > 0. \tag{6.41}
$$

Proof. On grounds of symmetry it is enough to prove only (6.40).

Let us substitute variables k, m with vaiables x, y by formulas $k = 2p + px$, $m = p + py$ in $J_1\{(2p,p),(2p,2p),(3p,p)\}$ and by formulas $k = 4p + px$, $m = 3p + py$ in $J_1\{(4p,3p),(4p,4p),(5p,3p)\}$, where $x, y \in (0,1), x + y < 1$, and consider the ratio of summands with equal values of x and y:

$$
\begin{aligned}
&\left|\frac{A(4p+px)A(3p+py)A(7p+px+py)}{A(2p+px)A(p+py)A(3p+px+py)}\right| \\
&= \left|\frac{\hat{D}(4+x)\hat{D}(3+y)\hat{D}(7+x+y)}{\hat{D}(2+x)\hat{D}(1+y)\hat{D}(3+x+y)}\right| \cdot e^{-Kt},
\end{aligned} \tag{6.42}
$$

where $K = 12p^2(5 + x + y) > 0$ for $x, y \in (0,1)$, and function $\hat{D}(x)$ is defined in (5.4).

For $t = 0$ the right-hand side of (6.34) has the form

$$
F(x,y) := f_1(y)f_2(y)f_3(x+y), \tag{6.43}
$$

where, according to (5.4),

$$
\begin{aligned}
f_1(x) = \frac{\hat{D}(4+x)}{\hat{D}(2+x)} &= \frac{(4+x)((2+x)^2-4)(16-(2+x)^2)}{((4+x)^2-4)((4+x)^2-16)(2+x)} \\
&= \frac{(4+x)^2}{(2+x)^2}\cdot\frac{2-x}{x+8}
\end{aligned} \tag{6.44}
$$

$$f_2(x) = \frac{\hat{D}(3+y)}{\hat{D}(1+y)} = \frac{(3+y)(4-(1+y)^2)(16-(1+y)^2)}{((3+y)^2-4)(16-(3+y)^2)(1+y)} = \frac{(3+y)^2}{(1+y)^2} \cdot \frac{(3-y)}{(7+y)} \tag{6.45}$$

$$f_3(z) = \frac{\hat{D}(7+z)}{\hat{D}(3+z)} = \frac{(7+z)((3+z)^2-4)(16-(3+z)^2)}{((7+z)^2-4)((7+z)^2-16)(3+z)} = \frac{(7+z)^2}{(3+z)^2} \cdot \frac{1-z^2}{(9+z)(11+z)} \tag{6.46}$$

Since every multiple of $f_1(x)$, $f_2(x)$ and $f_3(z)$ is decreasing,

$$F(x,y) < f_1(0)f_2(0)f_3(0) = \frac{7}{33}. \tag{6.47}$$

□

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